1-1-2014

(Visible) Tilings of Squares and Hypercubes

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(VISIBLE) TILINGS OF SQUARES AND HYPERCUBES

THESIS

A Thesis
presented in partial fulfillment of requirements
for the degree of Master of Science
in the Department of Mathematics
The University of Mississippi

by

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April 25, 2014

ABSTRACT

More than eighty years ago, Erdős considered sums of the side lengths of squares packed into a unit square. In particular, for n squares packed into a unit square, with side lengths $s_1, s_2, ..., s_n$, he set $f(n) = max\,(\Sigma s_i)$ where the max is taken over all packings with exactly n tiles. In 1995, he reminded us [1] of an unsolved conjecture: $f(k^2+1) = k$ for all k. In [2] Erdős and Soifer showed by construction that $f(n) > \sqrt{n-1}$ for all n except possibly $n = k^2+1$. In their constructions, the packings are in fact tilings. Here we consider various classes of tilings , this is, packings where there is no empty space inside the unit square. Several types of questions will be explored here. Various construction techniques are introduced, especially methods of generating tilings from tilings with fewer tiles. For some small values of n, I determine all tilings of the unit square with n tiles. I have found a best possible upper bound for $\sigma(T)$ where T is a visible tiling, that is a tiling which every tile shares a face with the unit square. Furthermore, I generalize this result to higher dimensions for visible tilings.

DEDICATION

I would like to dedicate this to my friends, family, and beautiful wife Tabitha.

ACKNOWLEDGEMENTS

I would like to thank Dr. William Staton and Dr. Benton Tyler for their contributions to this thesis and research over the years because without the help and guidance over the years this would not be possible. I look forward to working with you both in the future.

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1 INTRODUCTION

This thesis work begain with the author's interest in the following conjecture of Erdős from the 1930's:

"If $n = k^2 + 1$ squares tiles are packed into a unit square, the sum of the side lengths is less than or equal to k. "

No progress appeared until Erdős resuscitated the conjecture in the 1990's [2]. With Soifer, he reported some insights but no concrete progress [3]. Staton and Tyler reported slight progress with others have altered setting to triangles,rectangles and higher dimensional cubes [4]. Here, the focus on sums of side lengths of squares tiling a unit square is applied to visible tilings and very satisfactory bounds are obtained. Then the focus shifts to visible tilings in higher dimensions. Several issues arising in the course of the research led to definitions of new parameters. Most prominently, the 1903 theorem of Max Dehn [1]that in a tiling of a unit cube, every tile must have rational side lengths, led to the function $\Delta : \mathbb{N} \neq \{2, 3, 5\}$ = smallest d, such that a unit square can be tiled with n squares whose side lengths have least common denominator d.

In order to utilize the computational power of $Mathematica$, a tiling of the unit square with tiles of side length $s_1, s_2, s_3....., s_n$, with the least common denominator d, is replaced by tiling a $d_{s_1}, d_{s_2}, ..., d_{s_n}$ of a $d \times d$ square, with tiles of integer side lengths. Now, the equation $\sum_{i=1}^{n}(s_i)^2 = 1$ is replaced with $\sum_{i=1}^{n}(d_{s_i})^2 = d$. Hence the $(d_{s_i})^2$ constitute a partition od d^2 into n parts, and we can employ $Mathematica$ to generate partitions, which can be treated as candidates for areas of tiles.

Definition 1.0.1. To $tile$ a square is to fill it with squares, no two of which have an interior point in common.

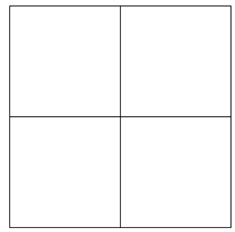


Figure 1.1: Example of Tiling T_1

Definition 1.0.2. A tiling T_2 is said to be a *simple subdivision* of tiling T_1 provided that T_2 is obtained from T_1 by subdividing a single tile of T_1. Tiling T_3 is said to be a subdivision of T_1 if T_3 is obtained by a finite sequence of simple subdivisions beginning with T_1.

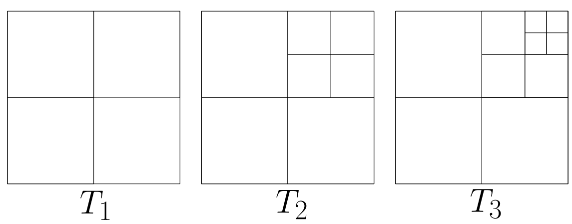


Figure 1.2: Example of Tiling T_2, where T_2 is a simple subdivision of T_1, while T_3 is a non-simple subdivision of T_1

Definition 1.0.3. A tiling with n tiles is *s-irreducible* if it is not a subdivision of any tiling with k tiles for any k between 2 and n-1.

Note that T, a tiling with *n* tiles of side lengths $s_1, s_2, s_3....., s_n$ the sum $\sum_{i=1}^{n}(s_i)^2 = 1$.

Definition 1.0.4. If T is a tiling of a unit square with n tiles of side lengths $s_1, s_2, s_3....., s_n$ we define $\sigma(T) = \sum\limits_{i=1}^{n} s_i$.

2 METHODS OF SUBDIVISON

In order to subdivide a square into smaller squares, it is important to look at many of the different types of constructions for tilings squares. Using the fact that all corners must be accounted for by only one square and that only edges may overlap, the construction of the squares can use multiple methods in order to give the desired rendering. Once we begin exploring these methods of tilings, we will combine them to generate other combinations of constructions.

2.1 TILINGS FOR A DESIRED NUMBER OF TILES

In this section we ask ourselves for what k there exists with exactly k tiles. There are many techniques to address this question. One of the first ways is to look at a number in *modulo 3 and apply the following to methods to reach the desired number. Mod 3 was chosen due to the process of taking one tile away adding four of equal size, thus gaining a net of 3 tiles.*

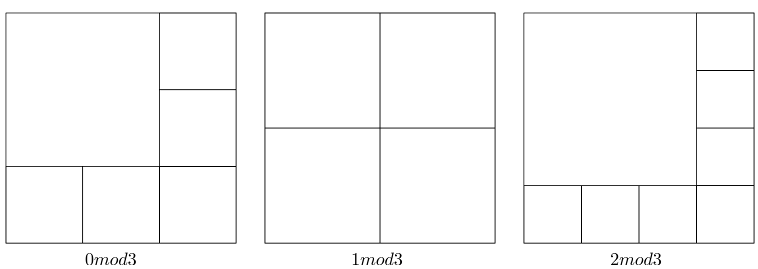


Figure 2.1: The smallest possible tilings of the respected module

The next two methods are basic methods used to derive all minimum modulo tilings as shown in 2.1 , as well as great tool to be able to take a smaller tiling and increase the number of tiles to reach the desired amount.

2.1 FIRST METHOD: (k - 1 + n^2) or "GRID" METHOD

This states that for any square tiled with k squares, then a square can be tiled by taking the initial collection of squares, choosing one, and replacing that square with n^2 equal sized squares, as shown in the Figure 2.1.2.

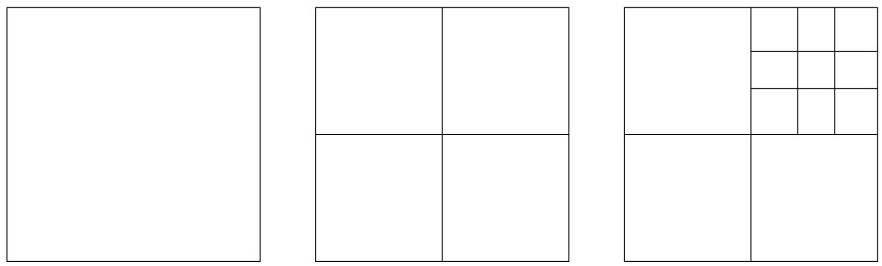
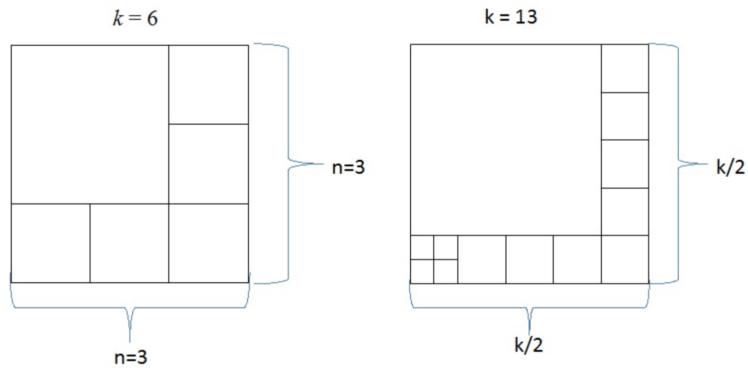


Figure 2.2: An example of starting with $k = 1$, then finding $k = (1 - 1 + 2^2) = 4$, then taking one of the 4 squares and applying the grid method to illustrate a tiling with $k = (4 - 1 + 3^2) = 12$ squares.

2.1 SECOND METHOD: n CUT METHOD



Case 1 (k is even): For any $k \geq 4$: The number of squares n, on each side of the the "L", will each be half of k for any desired number of tiles. For this case, all tilings will be irreducible.

Case 2 (k is odd): For any $k \geq 7$. Take $k - 3$, then place half of the remaining squares on each side of the "L". Then take one of the constructed squares and cut it into fourths to create it into four smaller squares as an application of the grid method.

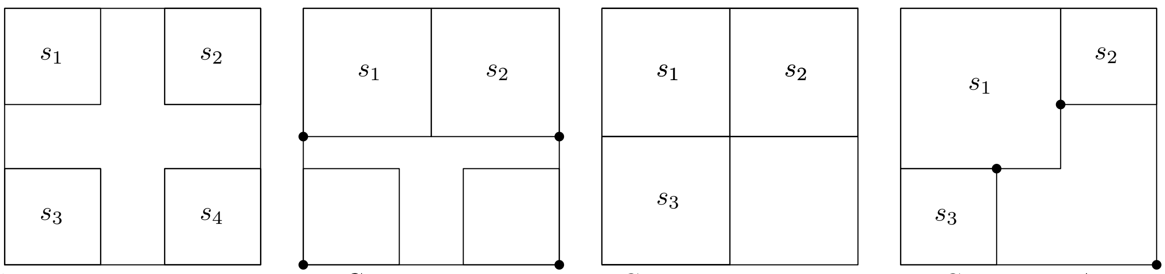
The previous method generates tilings of squares when applied to two sides. Using this method more then once can also produce similar results with all four sides being tiled with smaller squares.

I am going to begin using these techniques to prove a folk theorem.

Theorem 2.1.1. *A square can be tiled with n (non-degenerate) square tiles for all $n \neq 2, 3, 5$.*

Proof. Tilings with $n = 1, 4, 6, 7, 8$ are displayed in the figure [place me here]. By the $mod3$ construction above, any construction for k tiles will induce a construction for $k + 3$ tiles. Hence the existence for tiling $n = 6, 7, 8$ implies the existence for all tilings $k \geq 6$. A tile which covers more than one corner covers the whole unit square, hence this is possible only when $n = 1$. Therefore the tiles for $n = 2, 3$ are impossible due to a single tile would have to cover more than one corner which is impossible.

For $n = 5$, we will consider the proof in cases. Suppose there is a tiling with 5 tiles, with tiles of size s_1, s_2, s_3, s_4 covering the four corners, as shown below. If no pair of corner tiles meet (as in Case 1), there are four regions which must be cornered by additional tiles, which is impossible for $n = 5$. So suppose $s_1 + s_2 = 1$ (Case 2), then four corners of the remaining uncovered must be covered by 3 tiles, so one of those must cover 2 corners. This is impossible as a single square to be covered by two tiles is proven for $n = 2$. Hence $s_3 = \frac{1}{2}$ as shown in Case 3. This leaves a square to be covered with 2 squares which is impossible. Therefore $s_1 > s_2$. Now for Case 4, where $s_3 = s_2 = 1 - s_1$. The highlighted corners must be covered by three distinct tiles which is a contradiction due to we would have 6 tiles. Therefore a tiling for $n = 5$ is impossible.



Case 1:$s_1 = s_2 = s_3 = s_4$ Case 2: $s_1 = s_2$ Case 3:$s_1 = s_2 = s_3$ Case 4: $s_1 \neq s_2$

$\square$

2.1 UNIT FRACTIONS

We will now look at cases where the number 1 is expressed as the sum of unit fractions, such as $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$*. That is, each fraction being summed has 1 as a numerator and a denominator that is a positive integer. For all the fractions that we consider, the denominators differ from each other. This can show a unique tiling for a maximum expansion and* l *is the unit length of 1.*

For each n*, the natural numbers that will appear in the denominators of the* n *unit fractions that sum to 1, where each denominator is distinct. While other combinations of fractions can be derived, this table illustrates examples of 3 to 12 distinct unit fractions whose sum is 1. The following formula is useful for expressing unit fractions in terms of other distinct unit fractions.*

$$\frac{1}{m} = \frac{m+1}{m(m+1)} = \frac{m}{m(m+1)} + \frac{1}{m(m+1)} = \frac{1}{m+1} + \frac{1}{m(m+1)}$$

For example, setting $m = 2$:

$$\frac{1}{2} = \frac{1}{2+1} + \frac{1}{2(2+1)}$$

Another result for $m = 3$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

Thus can be shown from the example above, by completing another iteration :

$$\frac{1}{2} = \left(\frac{1}{4} + \frac{1}{12}\right) + \frac{1}{6}$$

Which when rearranged can derive $n = 4$. We can use the same formula to derive any following n. There exists a tiling for both squares and cubes using this method, although the number used in cubic tilings must be calculated slightly differently.

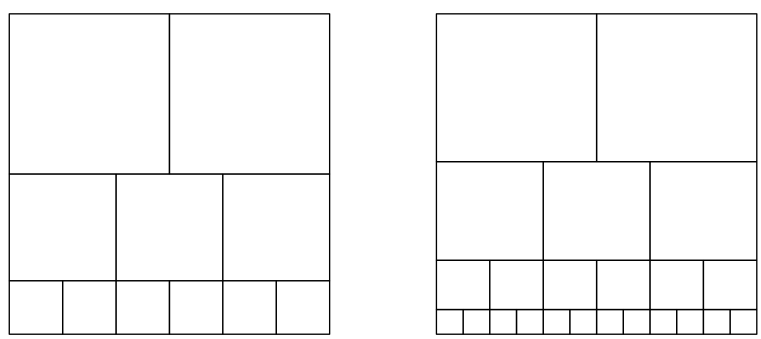


Figure 2.3: The first two applications of unit fractions

3 GENERATING TILINGS BY COMMON DENOMINATOR

An important result for my investigation is the following theorem of Max Dehn (1903).

Theorem 3.0.2 (Dehn)**.** *Every tile in a tiling of the unit square has rational side length.*

*The process of tiles sharing a common denominator was used when we began looking at computation software programs. We wanted to find a single denominator and find all possible constructions with that denominator. In doing so it is focused less on the amount of tiles initially but will generate multiple constructions from the same denominator. We combine both methods into Mathematica to begin to prove possible tilings for a given denominator d **and** number of tiles k.*

It will be convenient to classify tilings by the least common divisor of the tile lengths. Once this is done, it will be a further convenience to identify tilings of a unit square with tiles of side lengths of denominator d with the more convenient tilings of a $d \times d$ square with tiles of integer side lengths.

3.1 PROGRESS

Even with fixed denominator d and fixed number n of tiles, finding all tilings is computationally daunting. I have sought assistance from $Mathematica$.Using the fact that $\sum_{i=1}^{n}(s_i)^2 = 1$, and shifting to the convention of tiling a $d \times d$ square, we get $\sum_{i=1}^{n}(s_i)^2 = d^2$. The $IntegerPartition$ command in $Mathematica$ produces an output of all partitions of a given integer (d^2) into n parts, where the parts are chosen from a specified list (perfect squares). This program is implemented with $d = 10$ and $n = 8$. The output consists of all partitions of 100 into 8 perfect squares, each of which represents the area of a tile in a possible tiling of a $d \times d$ square. Now we process the output *further by taking square roots of the perfect squares to get a list of lists of possible tile sizes; each in descending order.*

In [1] : = IntegerPartitions [81, $\{$10,10$\}$, $\{$1,4,9,16,25,36,49,64$\}$]

Out [1] : =

$$\{\{64,9,1,1,1,1,1,1,1,1\}, \{49,16,9,1,1,1,1,1,1,1\}, \{49,9,4,4,4,4,4,1,1,1\},$$

$$\{36,25,4,4,4,4,1,1,1\}, \{36,16,4,4,4,4,4,4,4,1\}, \{36,9,9,9,4,4,4,4,1,1\}$$

$$\{36,16,16,4,4,1,1,1,1,1\}, \{25,25,16,9,1,1,1,1,1,1\}, \{25,25,9,4,4,4,4,4,1,1\},$$

$$\{25,16,9,9,9,9,1,1,1,1\}, \{25,16,9,9,9,9,1,1,1,1\}, \{25,16,9,9,4,4,4,1,1,1\},$$

$$\{16,16,16,9,4,4,1,1,1,1\}, \{16,16,16,9,4,4,1,1,1,1\}, \{16,16,16,9,4,4,4,4,4,4\},$$

$$\{16,9,9,9,9,9,9,9,1,1\}\}$$

We have now produced a list of candidates for tiles sizes for tilings of a 9×9 square with 10 tiles.

Lemma 3.1.1 (Burt,Staton)**.** *If T is a tiling of a $d \times d$ square with tile sizes ($s_1, s_2..., s_n$), then $s_1 + s_2 \leq d$.*

In [1] : = IntegerPartitions [81, $\{$10,10$\}$, $\{$1,4,9,16,25,36,49,64$\}$]

Out [1] : =

$$\{\{\cancel{64,9,1,1,1,1,1,1,1,1}\}, \{\cancel{49,16,9,1,1,1,1,1,1,1}\}, \{\cancel{49,9,4,4,4,4,4,1,1,1}\},$$

$$\{\cancel{36,25,4,4,4,4,1,1,1}\}, \{\cancel{36,16,4,4,4,4,4,4,4,1}\}, \{\cancel{36,9,9,9,4,4,4,4,1,1}\},$$

$$\{\cancel{36,16,16,4,4,1,1,1,1,1}\}, \{\cancel{25,25,16,9,1,1,1,1,1,1}\}, \{\cancel{25,25,9,4,4,4,4,4,1,1}\},$$

$$\{\cancel{25,16,9,9,9,9,1,1,1,1}\}, \{25, 16, 4, 9, 9, 9, 1, 1, 1, 1\}, \{\cancel{25,16,9,9,4,4,4,1,1,1}\},$$

$$\{\cancel{16,16,16,9,4,4,1,1,1,1}\}, \{\cancel{16,16,16,9,4,4,1,1,1,1}\}, \{\cancel{16,16,16,9,4,4,4,4,4,4}\},$$

At this point, we know that there is one list of possible tile sizes for a given denominator. Experimenting with the following constructions, I found several tilings with 9×9 *of 10 tiles. Two of these are displayed in Figure 3.1.*

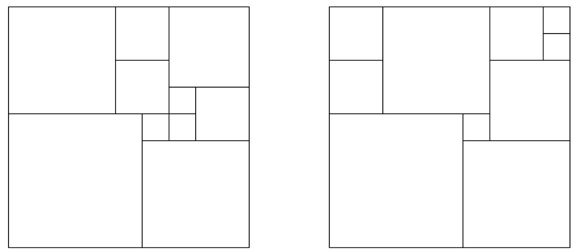


Figure 3.1:

4 RELATIONSHIP BETWEEN n, d

In finding all tilings for small values of d *and* n*, I noticed that* n *tends to be larger then* d*. The smallest example of* $d > n$ *occurs when* $d = 12$ *and* $n = 11$*; shown in Figure ref12with11. In fact, the relationship between* d *and* n *can be quite chaotic. On the one hand,* n *may be as large as* d^2*, as in the standard* $d \times d$ *grid. On the other hand, I have found tilings, described below, where* n *is only a small constant multiple of* $\sqrt{d}$*. The extremal cases would be interesting to know, for then we would be able in our* $Mathematica$ *searches to know how large the values of* d *must be considered in order to find all tilings for a fixed value of* n*.*

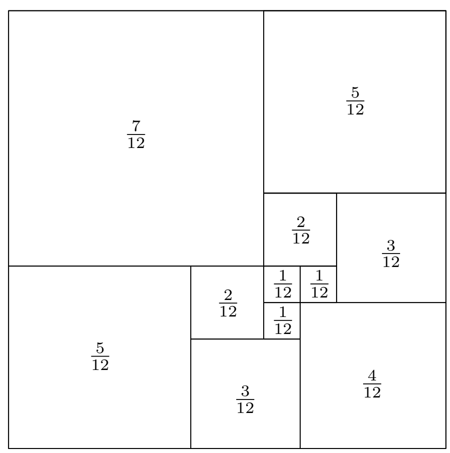


Figure 4.1: The first $s, d - irreducible$ tiling where $d > n$.

Here are general constructions for odd values of d*, showing that in fact* d *can be arbitrarily much larger than* n*.*

Definition 4.0.2. A tiling of a $d \times d$ square is said to be *d-irreducible* if the greatest common divisor of the tile sizes is 1.

Note that $d-irreducible$ and $s-irreducible$ are independent concepts. Figure 4.2 below shows two tilings each of which enjoys one of the two types of irreducibility but not the other.

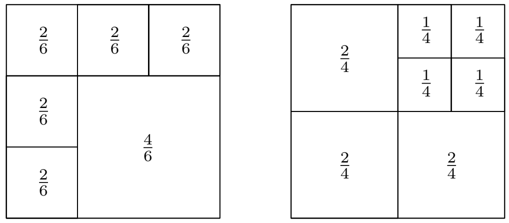


Figure 4.2:

Note that a tiling of a $d \times d$ square with at least one tile of side length 1 is always $d-irreducible$.

Now, several constructions will be shown in which d grows much faster than n. The first construction is very simple, but it shows that in a $d-irreducible$ the number of tiles may be as small as a constant times the logarithm of d.

<u>Construction 1</u>*: Let $d \geq 2$ be a positive integer. Let T_1 be the $a \times a$ grid with $d = a$ and $n = a^2$. If T_k is given, let T_{k+1} be obtained by subdividing one tile of T_k with an $a \times a$ grid. The first few tilings for $a = 2$ are shown in Figure 4.3. Tiling T_k has $d = a^k$ and $n = (a^2-1)log_a(d)+1$. Note that the T_k tilings are $d-irreducible$, but clearly not $s-irreducible$ since they are obtained by subdivisions.*

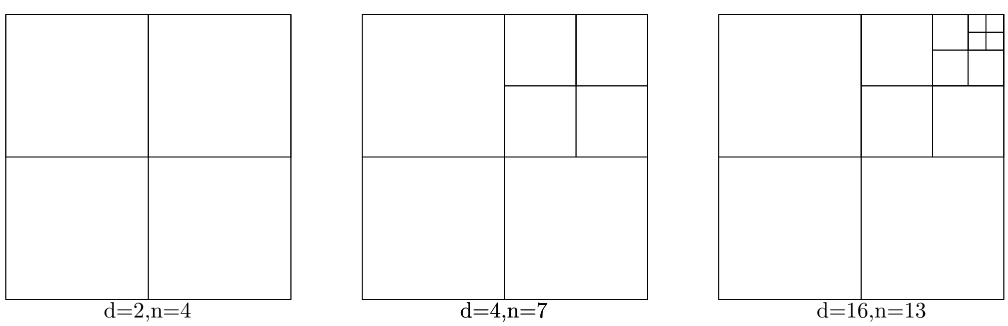


Figure 4.3:

<u>Construction 2</u>: Let $d \geq 11$ be an odd integer. Refer to Figure 4.4.

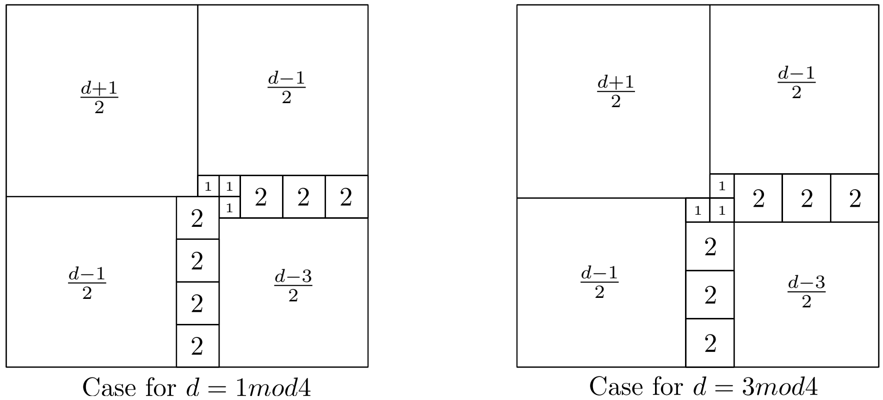


Figure 4.4:

Figure 4.5 representing cases for both constructions dealing with odd denominators.

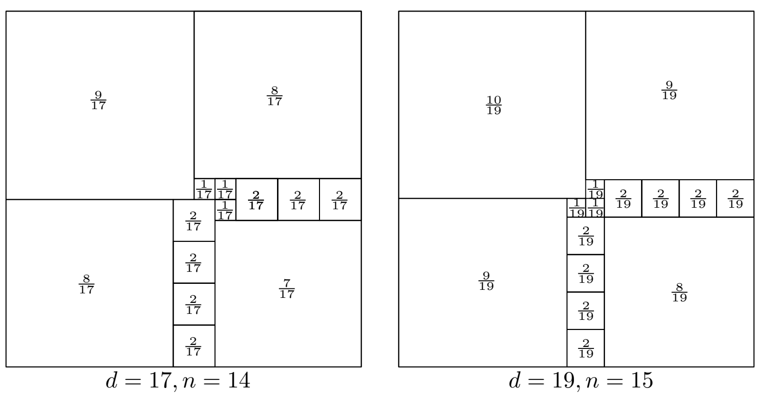


Figure 4.5:

In case, $n = 1mod4$, there are $\frac{d-5}{4}$ tiles of size 2 in the horizontal row and $\frac{d-1}{4}$ such tiles in the vertical column.

In case $n = 3mod4$, the horizontal row and the vertical row have $\frac{d-3}{4}$ such tiles.

Hence both cases have the following statistics.

Tile Size	Number of tiles
$\frac{d+1}{2}$	1
$\frac{d-1}{2}$	2
$\frac{d-3}{2}$	1
2	$\frac{d-3}{2}$
1	3

Note that Construction 2 is both $s, d - irreducible$. The total number of tiles $\frac{d+11}{2} = n$. Hence, even with both types of irreducibility, n may be as roughly small as half of d. Construction 2 yields tilings with n approximately $\frac{1}{d}$. Similar constructions with the strips of tiles of side length 3,4 yield tilings with $n \approx cd$ with $c < \frac{1}{2}$ and decreasing as the width of the strip increases. Rather than displaying the cases of strip width (3,4,.....) , I decided to try strips of size roughly $\sqrt{d}$ to see whether a small asymptotic value of n could be achieved.

Construction 3: For purposes of convenience, I will consider very special cases of d. In particular, let $d = 1 + k^2$ where k = 3 mod 6 and $k \geq 9$. Begin with four corner tiles as in Figure 4.6.

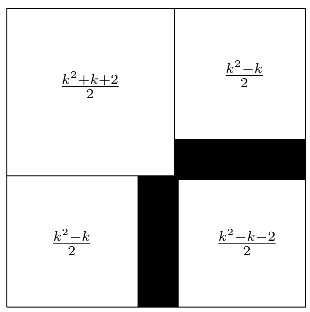


Figure 4.6:

In the shaded area, displayed below in Figure 4.7, add k-3 tiles of side length $k + 2$.

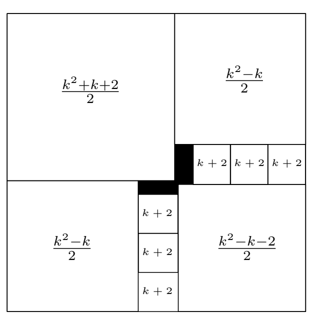


Figure 4.7:

Now add $\frac{2k+4}{3}$ tiles of size as shown in Figure 4.8.

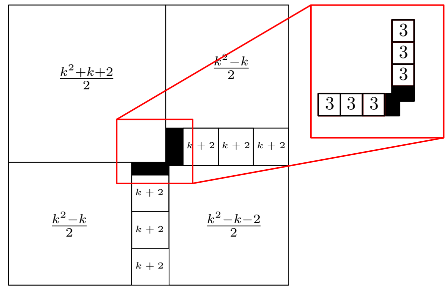


Figure 4.8:

Finally, add 5 tiles as follows in Figure 4.9:

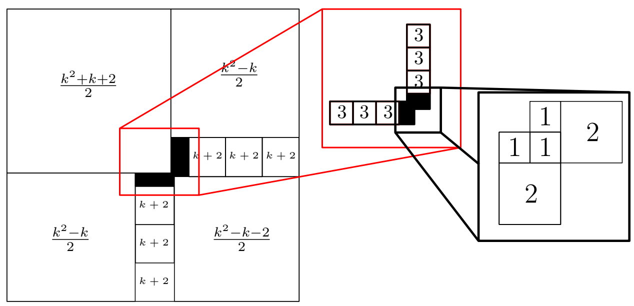


Figure 4.9:

The statistics for this tiling are as follows:

Tile Size	Number of tiles
$\frac{k^2+k+2}{2}$	1
$\frac{k^2-k}{2}$	2
$\frac{k^2-k-2}{2}$	1
$k+2$	$k-3$
2	2
1	3
3	$\frac{2k+4}{3}$

Eliminating k from the statistics yields

$$n = \frac{5\sqrt{d-1}+22}{3} \approx \frac{5}{3}\sqrt{d}$$

.

Below is picture for Figure 4.10 is the case of Construction 3 with $k = 99$. Here there are 171 tiles and $d = 9802$.

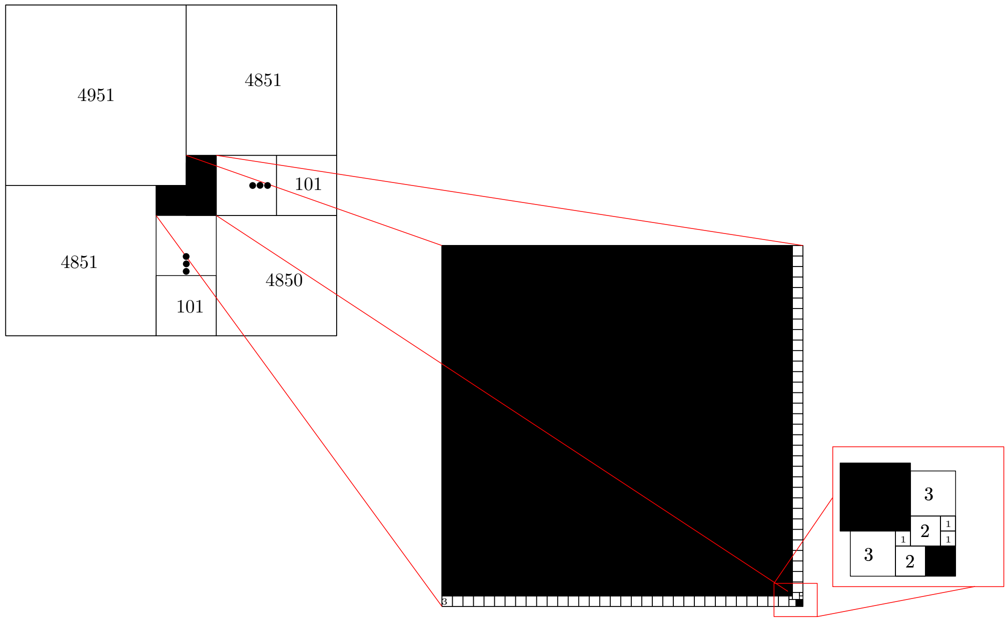


Figure 4.10: A scaled example of $d = 9802, n = 171$

Definition 5.0.3. To $tile$ a d-dimensional hypercube is to fill it with d-dimensional hypercubes, no two of which have an interior point in common.

Note that when tiling a d-dimensional unit hypercube with n tiles of side lengths s_1, s_2, s_3, ..., s_n; the sum $\sum_{i=1}^{n} s_i^d = s_1^d + s_2^d + s_3^d + ... + s_n^d$ will equal 1.

Definition 5.0.4. A $visible\ tiling$ of a d-dimensional hypercube is a tiling in which each tile has at least one face which is contained in a face of the hypercube being tiled. In other words, each tile in a visible tiling of a unit hypercube will have at least one surface with at least one coordinate of 0 or 1. Figure 5.1 illustrates examples of 2-dimensional visible tilings. Figure 5.2 illustrates examples of 2-dimensional tilings that are not visible.

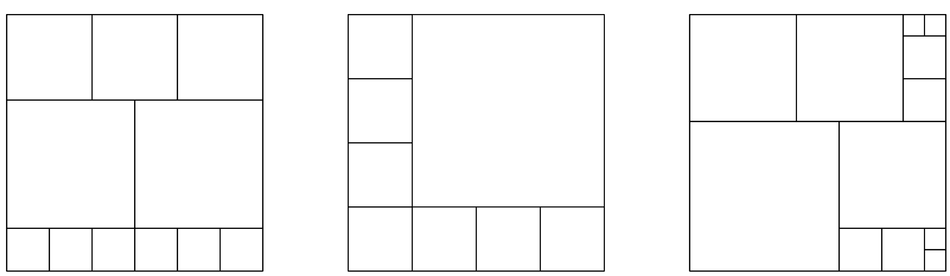


Figure 5.1: Sample visible tilings in 2 dimensions

Theorem 5.0.5 (Burt,Staton,Tyler)**.** *In a visible tiling of a d-dimensional hypercube, there will be at least two tiles of side lengths $1/2$ or at most one tile with side length greater than $1/2$.*

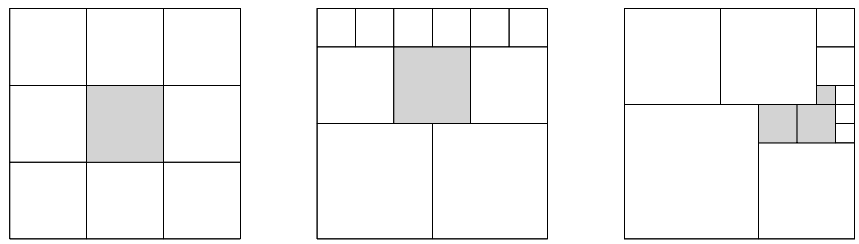


Figure 5.2: Sample tilings in two dimensions that are not visible (due to the presence of the shaded squares)

Proof. Consider the point in the center of the d-dimensional unit hypercube with coordinates $\left(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. Since each point in a tiling must be covered, this point must be covered by at least one hypercube.

Case 1: If this point is covered by a tile of side length $\frac{1}{2}$, then this tile's opposite side must lie on the boundary of the unit hypercube. Since the point will also need to be covered from the opposite side, a second tile of size $\frac{1}{2}$ must also be present.

Case 2: If the point is covered by a tile of side length greater than $\frac{1}{2}$, then each other tile in the unit hypercube must be smaller than this. □

Theorem 5.0.6 (Burt,Staton,Tyler)**.** *In any visible tiling of a square, there will be at least two squares of the smallest side length.*

Proof. Since we are dealing with a visible tiling, every tile must be either in a corner or along one of the edges. Consider the smallest tile present and the two associated cases:

Case 1: A smallest tile c rests in a corner as in Figure 5.3. If tiles a and b are the same size as tile c then we are done. Otherwise, either tile a or tile b must be larger than tile c. Without loss of generality, if tile a is bigger than tile c, then tile b can be at most the same size as tile c. Therefore, at least two of these tiles will be the same size.

Case 2: A smallest tile c sits along one of the edges as in Figure 5.4. If either tile a or b is the same size as tile c then we are done. Otherwise, tiles a and b must be larger than tile c. Since this is a tiling, this forces the rest of the area between each of these tiles to be occupied by another

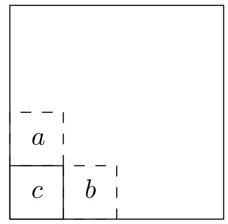


Figure 5.3: The smallest tile c is located at a corner

tile, call it d. Tile d must be as least as big as tile c, but no larger. Hence, tiles c and d must be the same size.

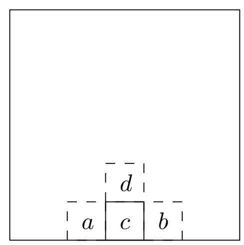


Figure 5.4: The smallest tile c is located along an edge

$\square$

We leave it to the reader to verify as a very interesting exercise that in a visible tiling of a cube there must be at least four cubes of the smallest size present.

Definition 5.0.7 (Burt,Staton,Tyler)**.** In this paper, we will consider a parameter, $\psi_d(n)$, that we will define as the maximum of s_1^{d-1} + s_2^{d-1} + s_3^{d-1} +...+ s_n^{d-1} among all visible tilings of a unit hypercube with n tiles in d dimensions.

Calculating s_1^{d-1} + s_2^{d-1} + s_3^{d-1} +...+ s_n^{d-1} *for a visible tiling of a* d*-dimensional unit hypercube can be done in the following way:*

- *For 2 dimensions, this sum is equal to* $4 - \Sigma c_i$*, where* c_i *are the side lengths of the corner tiles.*

- *In 3 dimensions, this sum is* $6 - \Sigma(e_i)^2 + \Sigma(c_i)^2$*, where each* e_i *is the side length of a cube along an edge, and is included once for each edge along which the cube appears. The* c_i *are the side lengths of all cubes that appear on the corners.*

- *This process can be extended into* d *dimensions, by using the principle of inclusion and exclusion.*

When calculating $\psi_2(n)$*, since corner squares appear on two sides, the näive approach of summing the side lengths along each edge counts the corner tiles twice, it is clear that* 4 *is a strict upper bound for* $\psi_2(n)$*. A similar argument shows that* 6 *is a strict upper bound for* $\psi_3(n)$*. More generally, we will show in Theorem 5.0.10 that* $2d$ *is a strict upper bound for* $\psi_d(n)$ *and that* $\lim_{n\to\infty} \psi_d(n) = 2d$*.*

Theorem 5.0.8 describes in detail how to obtain a value $\psi_2(n)$ *as close to 4 as desired.*

Theorem 5.0.8 (Burt,Staton,Tyler)**.** *Given any* α*, where* $\alpha > 0$*, there exists a tiling of a unit square, the sum of whose side lengths is greater than* $(4 - \alpha)$*, so that* $\psi_2(n) > (4 - \alpha)$*.*

Proof. Given any $\alpha > 0$, there exists some smallest even natural number k such that $\frac{4}{k} < \alpha$.

We will now consider a visible tiling of a unit square composed of the following three sizes of smaller squares:

1. $2k$ squares of side length $\frac{1}{k}$ arranged down the left and right sides of the unit square.

2. 1 square of side length $\frac{k-2}{k}$, in the middle and bottom of the unit square.

3. $\frac{k-2}{2}$ squares of side length $\frac{2}{k}$, resting on top of the larger square in (2) above.

Sample constructions for this process are illustrated in Figure 5.5 below.

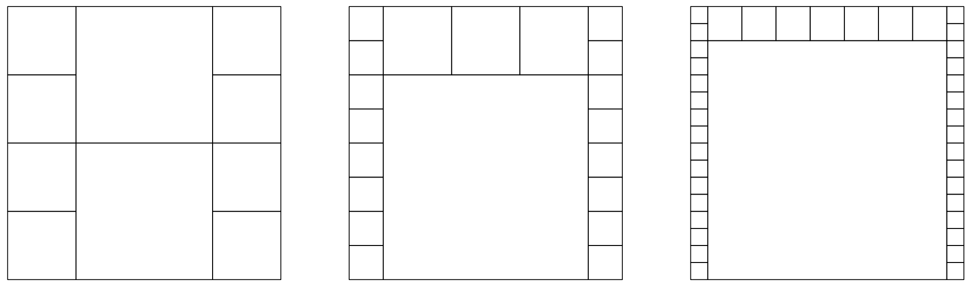


Figure 5.5: Sample tilings of squares for Theorem 5.0.8 in which $k = 4$, $k = 8$, and $k = 16$

Calculating the sum of the side lengths of the squares in this construction gives the following:

$$\begin{aligned}\psi_2\left(2k+1+\frac{k-2}{2}\right) &\geq (2k)\left(\frac{1}{k}\right)+(1)\left(\frac{k-2}{k}\right)+\left(\frac{k-2}{2}\right)\left(\frac{2}{k}\right)\\ &= 2+1-\frac{2}{k}+1-\frac{2}{k}\\ &= 4-\frac{4}{k} > 4-\alpha\end{aligned}$$

□

We will now prove a similar result in three dimensions using our definition for $\psi_3(n)$.

Theorem 5.0.9 (Burt,Staton,Tyler)**.** *Given any α, where $\alpha > 0$, there exists a visible tiling of a unit cube, one in which one sixth the sum of the surface areas is less than $(6-\alpha)$, so that $\psi_3(n) > (6-\alpha)$.*

Proof. Given any $\alpha > 0$, there exists some smallest even natural number k such that $\frac{12k-8}{k^2} < \alpha$.

We will now consider a visible tiling of a unit cube composed of the following three sizes smaller cubes:

1. $4(k-1)k = 4k^2 - 4k$ cubes of side length $\frac{1}{k}$ arranged around a center square prism.

2. 1 cube of side length $\frac{k-2}{k}$, in the middle and bottom of the unit cube.

3. $\left(\frac{k-2}{2}\right)^2$ cubes of side length $\frac{2}{k}$, resting on top of the larger cube in (2) above.

Sample constructions for $k = 4$ and $k = 8$ are illustrated in Figure 5.6.

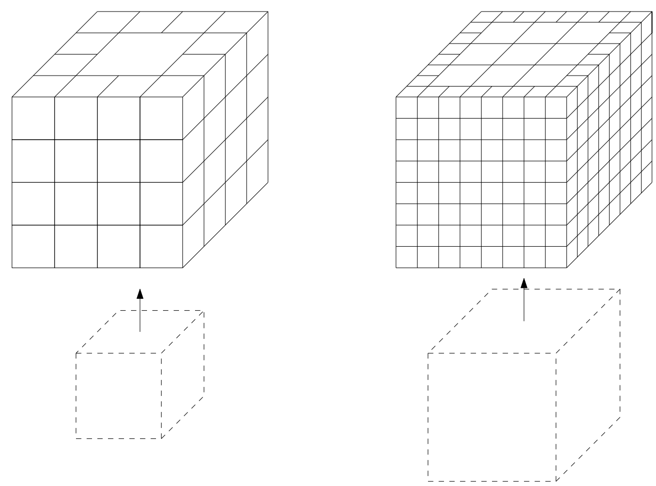


Figure 5.6: Sample tilings of cubes for Theorem 5.0.9 in which $k = 4$ and $k = 8$

Calculating the sum of one of the sides of each of the cubes in this construction yields:

$$\begin{aligned}
\psi_3\left(4k^2 - 4k + 1 + \left(\frac{k-2}{2}\right)^2\right) &\geq (4k^2 - 4k)\left(\frac{1}{k}\right)^2 + (1)\left(\frac{k-2}{k}\right)^2 + \left(\frac{k-2}{2}\right)^2\left(\frac{2}{k}\right)^2 \\
&= \frac{4k^2 - 4k}{k^2} + \frac{k^2 - 4k + 4}{k^2} + \frac{k^2 - 4k + 4}{k^2} \\
&= \frac{4k^2 - 4k + k^2 - 4k + 4 + k^2 - 4k + 4}{k^2} \\
&= \frac{6k^2 - 12k + 8}{k^2} \\
&= 6 - \frac{12k - 8}{k^2} > 6 - \alpha
\end{aligned}$$

$\square$

Figure 5.6 makes it apparent why our definition for $\psi_3(n)$ considers the surface areas of visible cubes tiling a unit cube, rather than simply the side lengths of the cubes since the sum of these side lengths does not remain bounded with larger values of k.

Theorems 5.0.8 and 5.0.9 follow from the more general Theorem 5.0.10 below but have been included for the sake of concreteness.

Theorem 5.0.10 (Burt,Staton,Tyler). *In d-dimensions, for any $\alpha > 0$, there exists a visible tiling, with so that $\psi_d(n) > 2d - \alpha$.*

Proof. We will prove this by generalizing our results for Theorem 5.0.8 and Theorem 5.0.9 above. Taking a d-dimensional hypercube, we will break it down into smaller hypercubes as follows:

1. 1 hypercube of side length $\frac{k-2}{k}$. One of the corners of this hypercube will have coordinates $\left(0, \frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}\right)$, with the opposite corner having coordinates $\left(1 - \frac{2}{k}, 1 - \frac{1}{k}, 1 - \frac{1}{k}, ..., 1 - \frac{1}{k}\right)$.
2. $\left(\frac{k-2}{2}\right)^{d-1}$ hypercubes of side length $\frac{2}{k}$. These hypercubes will occupy a region with one corner located at $\left(1 - \frac{2}{k}, \frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}\right)$ and the opposite corner located at $\left(1, 1 - \frac{1}{k}, 1 - \frac{1}{k}, ..., 1 - \frac{1}{k}\right)$.
3. $\left(k^d - (k-2)^d - \left(\frac{k-2}{2}\right)^{d-1}(2)^d\right)$ hypercubes of side length $\frac{1}{k}$ arranged between the hypercubes in (1) and (2) above and the boundary of the d-dimensional hypercube. Each of these smaller hypercubes will have at least 2^{d-1} corners with coordinates that contain a 0 or 1.

Verifying that $\sum_{i=0}^{n} s_i^d = 1$ for these hypercubes, we have:

$$(1)\left(\frac{k-2}{k}\right)^d + \left(\frac{k-2}{2}\right)^{d-1}\left(\frac{2}{k}\right)^d + \left(k^d - (k-2)^d - \left(\frac{k-2}{2}\right)^{d-1}(2)^d\right)\left(\frac{1}{k}\right)^d$$
$$= 1 - \left(\frac{k-2}{k}\right)^d - \frac{2(k-2)^{d-1}}{k^d} + \left(\tfrac{k-2}{k}\right)^d + \frac{2(k-2)^{d-1}}{k^d}$$
$$= 1$$

We will thus have 1 tile of side length $\frac{k-2}{k}$, $(\frac{k-2}{2})^{d-1}$ tiles of side length $\frac{2}{k}$, and $[k^d - (k-2)^d - (\frac{k-2}{2})^{d-1}(2)^d]$ tiles of side length $\frac{1}{k}$. In summing the $d-1$ powers of the tile side lengths, we have:

$$(1)\left(\frac{k-2}{k}\right)^{d-1}+\left(\frac{k-2}{2}\right)^{d-1}\left(\tfrac{2}{k}\right)^{d-1}+\left[k^d-(k-2)^d-\left(\frac{k-2}{2}\right)^{d-1}(2)^d\right]\left(\frac{1}{k}\right)^{d-1}$$

$$=\left(\frac{1}{k}\right)^{d-1}\left[(k-2)^{d-1}+\left(\frac{k-2}{2}\right)^{d-1}(2)^{d-1}+k^d-(k-2)^d-\left(\frac{k-2}{2}\right)^{d-1}(2)^d\right]$$

$$=\left(\frac{1}{k}\right)^{d-1}\left[(k-2)^{d-1}+(k-2)^{d-1}+k^d-(k-2)^d-2(k-2)^{d-1}\right]$$

$$=\left(\frac{1}{k}\right)^{d-1}\left[k^d-(k-2)^d\right]$$

$$=k-\frac{(k-2)^d}{k^{d-1}}$$

$$=k-\frac{\sum_{i=0}^{d}\binom{d}{i}^{d-i}k^{d-i}(-2)^i}{k^{d-1}}$$

$$=k-\frac{k^d+d(-2)k^{d-1}+\sum_{i=2}^{d}\binom{d}{i}^{d-i}k^{d-i}(-2)^i}{k^{d-1}}$$

$$=k-k-d(-2)-\frac{\sum_{i=2}^{d}\binom{d}{i}^{d-i}k^{d-i}(-2)^i}{k^{d-1}}$$

$$=2d-\frac{\sum_{i=2}^{d}\binom{d}{i}^{d-i}k^{d-i}(-2)^i}{k^{d-1}}$$

If we fix d, as k approaches ∞, this value approaches $2d$. Therefore, given any $\alpha > 0$, there exists a sufficiently large even natural number k such that this construction will give:

$$\psi_d\left(1+\left(\frac{k-2}{2}\right)^{d-1}+\left[k^d-(k-2)^d-\left(\frac{k-2}{2}\right)^{d-1}(2)^d\right]\right)$$
$$\geq 2d-\frac{\sum_{i=2}^{d}\binom{d}{i}^{d-i}k^{d-i}(-2)^i}{k^{d-1}} > 2d-\alpha.$$

□

6 ALL TILINGS FOR SMALL (n, d)

We know that for $n \neq 2, 3, 5$ there is a tiling of a square with n tiles. It is interesting to know for a given n, what is the smallest d for which a $d \times d$ square can be tiled with n tiles of integer side lengths.

Definition 6.0.11. For $n \neq 2, 3, 5$, $n \in \mathbb{N}$, $\Delta(n) =$ smallest d such that a $d \times d$ square can be tiled with n tiles of integer side lengths.

The following is obvious.

Theorem 6.0.12. *If $n > d^2$ then $\Delta(n) \geq d$.*

Proof. A $(d-1) \times (d-1)$ square cannot have d^2 tiles of integer side lengths.

□

Corollary 6.0.13. *For all $n \in \mathbb{N}$, $\Delta(n) \geq \lceil \sqrt{n} \rceil$.*

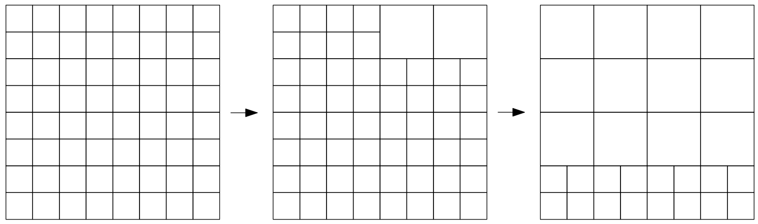
Conjecture 6.0.14. *For all $n \in \mathbb{N}$, $\lceil \sqrt{n} \rceil \leq \Delta(n) \leq \lceil \sqrt{n} \rceil + 2$.*

This conjecture holds for all values for $\Delta(n)$ computed. I believe there is a possibility with a finite number of counter examples that the upper bound can be reduced by 1.

Theorem 6.0.15. *If $(d-1)^2 < n < d^2$ and $d^2 = n mod 3$ then $\Delta(n) = d$.*

Proof. Note that since the number of tiles is strictly greater than $(d-1)^2$ then no tiling with n can be done with any denominator less than d. From the $d \times d$ square with d^2 tiles, remove $(\frac{d^2-n}{3})$ 2×2 squares, each containing 4 tiles. Replace each with a 2×2 tile.

□



For small values of n, I have used the Mathematica $IntegerPartition$ command to examine possible values of d for each n.

n	1	4	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Delta(n)$	1	2	3	4	4	3	4	5	6	4	5	6	4	5	6	5	6

Note that $\Delta(n)$ is not a nondecreasing function and that for $n = d^2$ we have $\Delta(n) = d$.

The following is a table listing the number of possible tilings as generated by $Mathematica$ with figures following for small $n = 6 - 12$.

n, d	Number of tilings	Psi Value
4,2	1	2
6,3	1	2.33
7,4	1	2.33
8,4	1	2.5
8,5	1	2.6
9,3	1	3
9,6	2	2.66; 2.83
9,7	1	2.71
10,4	1	3
10,5	1	2.6
10,7	2	2.71; 2.71
10,8	2	2.75; 2.75
10,9	1	2.78
11,5	1	3
11,6	1	3
11,8	2	2.75; 3.25
11,9	2	2.78; 3.22
11,12	1	2.83
12,6	4	3; 3; 3.17; 3.33
12,9	1	2.78
12,11	2	3.18;2.81
12,12	3	2.83; 2.83; 2.83

7 ALL POSSIBLE TILINGS FOR SMALL n

For the remainder for the text we will note that the following constructions are all d-irreducible.

7.1 $n = 6, 7, 8$

For the case of k=6, we also have this as a base case for the smallest case of $0 mod 3$ *possible in tilings. For* $k = 7$ *it is also the smallest case in which we can apply the grid method, also seen as imbedding* $k = 4$ *into a previous construction. The results for* $k = 8$ *were generated and confirmed by Mathematica.*

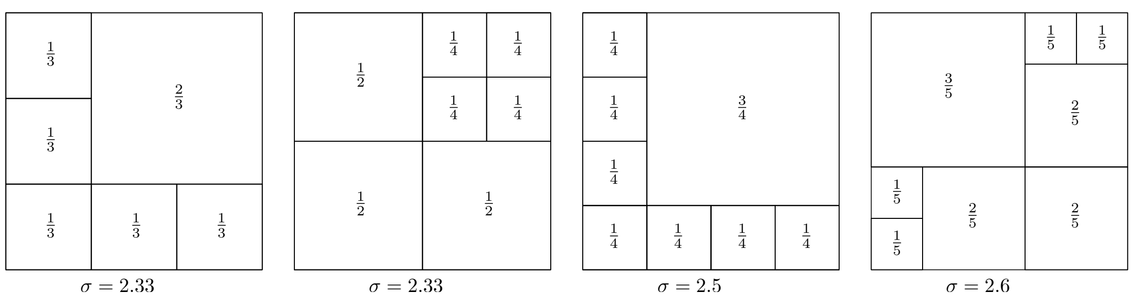


Figure 7.1: All possible

7.2 $n = 9$

The constructions for $n = 9$ *as confirmed by* $Mathematica$ *are also an interesting case as it is the first non-trivial perfect square as originally posed by Erdős.*

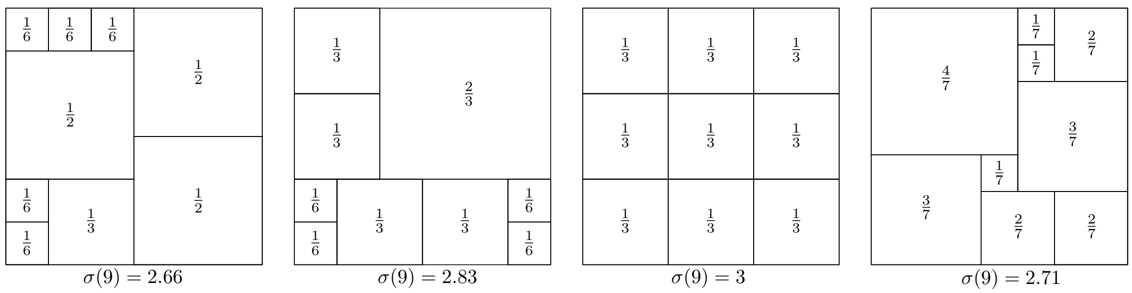
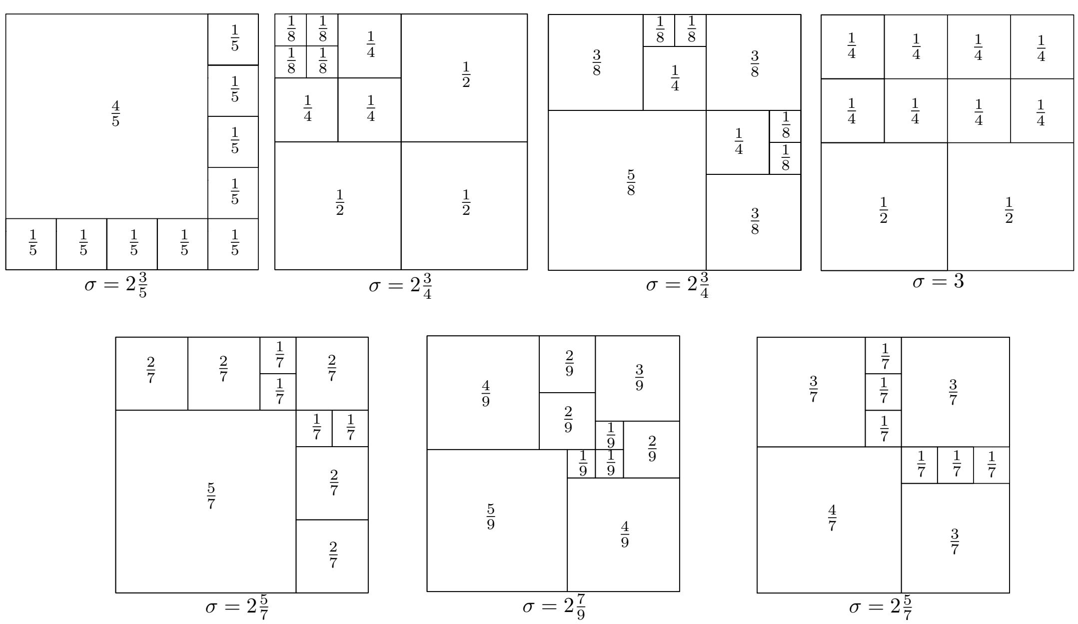


Figure 7.2: The third construction first non-trivial perfect square

7.3 $n = 10$

One of the first interesting questions and the original interest for this research was in hopes of finding the case Erdős posed when $\sigma(k^2 + 1) = \sigma(k^2)$. *After going through all output generated by Mathematica, the following figure are the possible d-irreducible tilings for* $k = 10$.



7.4 $k = 11$

The following constructions in the figure below represent all generated tilings as confirmed by $Mathematica$.

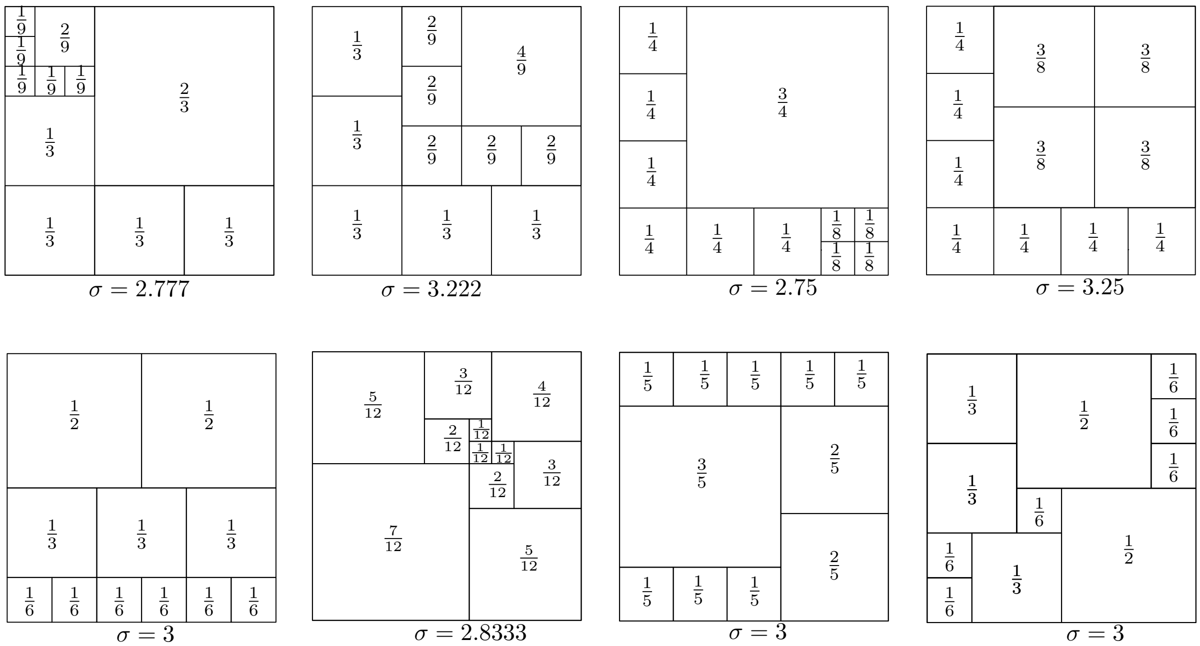
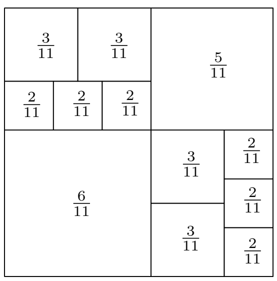


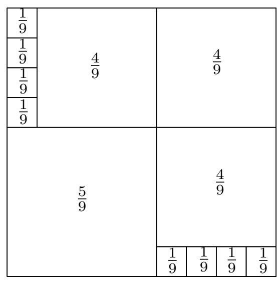
Figure 7.3: Constructions for 11

7.5 $k = 12$

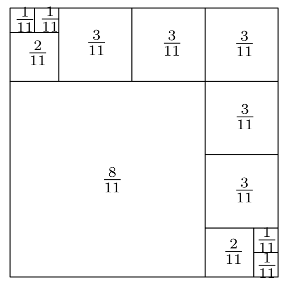
The optimal tiling for the sum of the size lengths for $k^2 + 2$ is trivial due to we can the the previous k^2 and play a single application of the "grid" method to find the maximum sum of the side lengths.



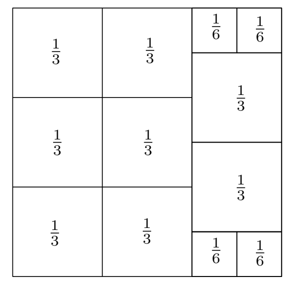
$\sigma = 3.1818$



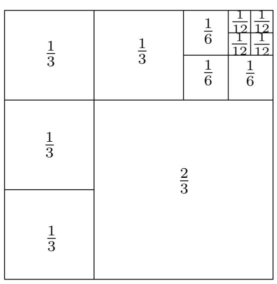
$\sigma = 2.7777$



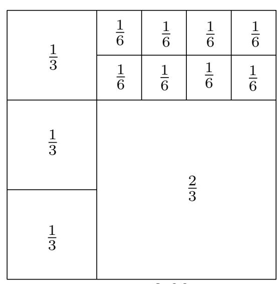
$\sigma = 2.8181$



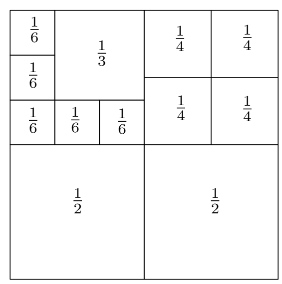
$\sigma = 3.333$



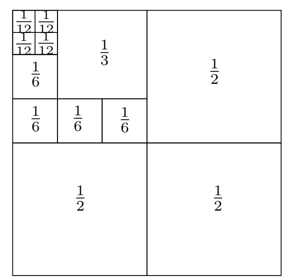
$\sigma = 2.833$



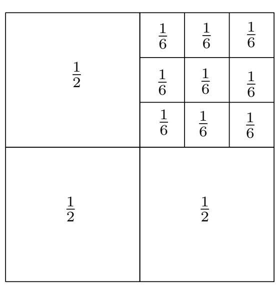
$\sigma = 3.00$



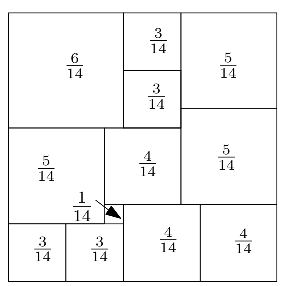
$\sigma = 3.1666$



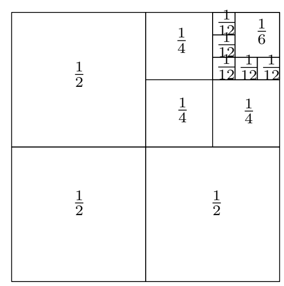
$\sigma = 2.8333$



$\sigma = 3$



$\sigma = 3.27$



$\sigma = 2.833$

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add the VITA to the Table of Contents

VITA

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On Visible Tilings -AMS Graduate Student Seminar